Outline of the Boundary Element Method

In this document the overall strategy in using the Boundary Element Method (BEM)\(^1\) is briefly outlined. For a more thorough introduction, the reader is advised to consult the textbooks exclusively devoted to the boundary element method\(^2\). In the derivation of the BEM, the underlying objective is to replace the partial differential equation that governs the solution in a domain by an equation that governs the solution on the boundary alone. For example the Laplace equation\(^3\)

$$\sum_{i=1}^{N} \frac{\partial^2 \varphi(p)}{\partial x_i^2} = 0$$

where \(N\) is the dimension of the space, or more concisely,

$$\nabla^2 \varphi = 0 . \tag{1}$$

In this outline of the BEM, we consider the interior problem\(^4\); Laplace's equation governs the interior to a domain \(D\) bounded by a boundary \(S\) as shown in the figure below

Illustration of the interior domain.

can be replaced by an integral equation (Fredholm)\(^5\) of the form

$$\int_{S} \frac{\partial G(p, q)}{\partial n_q} \varphi(q) dS_q + \frac{1}{2} \varphi(q) = \int_{S} G(p, q) \frac{\partial \varphi(q)}{\partial n_q} dS_q . \tag{2}$$

The function \(G\) is known as a Green's function\(^6\). Physically, \(G(p, q)\) represents the effect observed at a point \(p\) of a unit source at the point \(q\). The terminology \(\frac{\partial^*}{\partial n_q}\) represents the

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\(^1\) Boundary Element Method
\(^2\) Boundary Element Method texts
\(^3\) Laplace Equation
\(^4\) Boundary Value Problems and Boundary Conditions
\(^5\) Integral Equations
\(^6\) The Laplace Integral Operators
partial derivative\(^7\) of the function \(*\) with respect to the unit outward normal at the point \(q\) on the boundary.

The integral equation can be derived from the Laplace equation by applying Green's second theorem. The power of the formulation lies in the fact that it relates the potential \(\phi\) and its derivative on the boundary alone; no reference is made to \(\phi\) at points in the domain. In a typical boundary-value problem we may be given \(\phi(q), \frac{\partial \phi}{\partial n_q}\) or a combination of such data on \(S\): the boundary integral equation is a means of determining the unknown boundary function(s) from given boundary data.

**Operator Notation**

Operator notation is a useful shorthand in writing integral equations. Moreover, it will be shown that it is very powerful notation in that it clearly demonstrates the connection between the integral equation and the linear system of equations that results from its discretisation.

Integral equations can always be written in terms of integral operators. For example if \(\zeta\) is a function defined on a boundary \(\Gamma\) then applying the following operation to \(\zeta\) for all points \(p\) on \(\Gamma\)

\[
\int_{\Gamma} G(p,q) \zeta(q) dS_q = \nu(p) \quad (p \in S)
\]

gives a function \(v\). This can be viewed as the application of an operator to the function \(\zeta\) to return the function \(v\). More simply we may write

\[
\{L\zeta\}_\Gamma(p) = v(p). \quad (3)
\]

In equation (3) the \(L\) represents the integral operator and the subscript \((\Gamma)\) refers to the domain of integration. Here \(\Gamma\) is used as a variable, representing either a whole surface or a patch of surface.

In operator notation the integral equation (1) can be written in the alternative shorthand notation

\[
\{M\phi\}_\Gamma(p) + \frac{1}{2} \phi(p) = \{L\nu\}_\Gamma(p) \quad \text{or} \quad \left(\{M + \frac{1}{2}I\} \phi\right)_\Gamma(p) = \{L\nu\}_\Gamma(p) \quad (4)
\]

where \(\nu(q) = \frac{\partial \phi}{\partial n_q}\), the \(M\) represents the other integral operator;

\[
\{M\zeta\}_\Gamma(p) = \int_S \frac{\partial G(p,q)}{\partial n_q} \zeta(q) dS_q \quad (5)
\]

and \(I\) represents the identity operator.

**Numerical Solution of the Integral Equation**

In order to develop a numerical method for the solution of integral equations like (4), a technique is applied so that the equation is simplified into a linear system of equations\(^8\).

\(^7\) Partial Differentiation

\(^8\)
Hence there is a close analogy between linear integral equations and systems of linear equations; the integral operators can be viewed as matrices, the boundary functions as vectors. The application of such a technique transforms the equation (4) to an equation of the form

\[ M \hat{\varphi} + \frac{1}{2} \hat{\varphi} = L \hat{v} \text{ or } (M + \frac{1}{2}I) \hat{\varphi} = L \hat{v} \quad (6) \]

where the components of the vectors \( \hat{\varphi} \) and \( \hat{v} \) represent (approximations to) the values of the functions \( \varphi(p) \) and \( \frac{\partial \varphi(p)}{\partial n_q} \) at a set of points on the boundary. L, M and I are matrices\(^9\) derived from the corresponding integral operators in (4) with I representing the identity matrix. The connection between the system of linear equations (6) and the integral equation (4) or (2) is now clear. As stated earlier, the boundary data \( \varphi(p) \) and \( \frac{\partial \varphi(p)}{\partial n_q} \), or some combination of the two functions are given and the solution of the system of linear equations (6) can be used to derive approximations to the unknown boundary data. There are a variety of techniques for deriving the system of linear equations from a given integral equation. In general, a method can be derived by replacing the integrals in an integral equation by a quadrature formula or by a weighted residual method such as the Galerkin method. Many methods for solving integral equations can be used to develop a particular boundary element method. The method employed in the software accompanying this text is that of collocation\(^10\) since it is considerably easier to program and probably more efficient than competing techniques.

The application of collocation to a boundary integral equation requires that the boundary is represented by a set of panels\(^11\). For example a two dimensional boundary can be approximated by a set of straight lines as illustrated in the figure below. In order to complete the discretisation of the integral equations, the boundary functions also need to be approximated on each panel. In this work, it is the characteristics of the panel and the representation of the boundary function on the panel that together define the element in the boundary element method. By representing the boundary functions by a characteristic form on each panel, the boundary integral equations can be written as a linear system of equations of the form introduced earlier.

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\(^8\) Numerical Solution of Integral Equations
\(^9\) Matrices
\(^10\) Solution of Fredholm Integral Equations by Collocation
\(^11\) Boundary Representation in the Boundary Element Method
Domain Solution

On solution of the integral equation (approximations to) the unknown boundary function(s) will be known on \( S \). Hence both of \( \varphi(p) \) and \( \frac{\partial \varphi(p)}{\partial n_q} \) will be explicit if the method is based on the integral equation (6). In most cases a solution in the domain \( D \) is required and this can be found using the following equation:

\[
\varphi(p) = \int_S G(p, q) \frac{\partial \varphi}{\partial n_q} dS_q - \int_S \frac{\partial G(p, q)}{\partial n_q} \varphi(q) dS_q \quad (p \in D)
\]

or, more concisely,

\[
\varphi(p) = \{L \varphi\}_S - \{M \nu\}_S \quad (p \in D) \tag{7}
\]

using the notation introduced earlier. The domain solution \( \varphi(p) \) for any point \( p \) in the domain can be obtained through simply evaluating the integrals in (7).

Direct and Indirect Boundary Integral Equations

There are two fundamental approaches to the derivation of an integral equation formulation of a partial differential equation. The first is often termed the direct method; the integral equations are derived through the application of Green’s second theorem and an example of this has already been given earlier. The other technique is termed the indirect method. This is based on the assumption that the solution can be expressed in terms of a source density function defined on the boundary. For example it is assumed that the solution of the Laplace equation can be written in the form
\[ \varphi(p) = \int_S G(p, q) \sigma(q) dS_q , \]

where \( \sigma \) is the source density function defined on \( S \) only. The following integral equation can be derived from the above by differentiating in the direction of the normal to the boundary:

\[ \frac{\partial \varphi(p)}{\partial n_p} = \frac{\partial}{\partial n_p} \int_S G(p, q) \sigma(q) dS_q + \frac{1}{2} \sigma(p) = \int_S \frac{\partial G(p, q)}{\partial n_p} \sigma(q) dS_q + \frac{1}{2} \sigma(p) \]

In operator notation the above integral equations are written

\[ \varphi(p) = \{ L \sigma \}_S(p) \quad \text{and} \quad v(p) = \{ (M^t + \frac{1}{2} I) \sigma \}_S(p) \quad (p \in S). \]

Note the relationship between the operator \( M^t \) and the operator \( M \) introduced earlier; \( M^t \) is known as the transpose of \( M \) and is arrived at simply by swapping the arguments \( p \) and \( q \) in the definition.

In some cases the indirect method is a more versatile reformulation of the partial differential equation as the terms \( \varphi \) and \( \frac{\partial \varphi}{\partial n} \) are already isolated; the expressions for can be substituted directly into the boundary condition, giving the indirect integral equation formulation. For a general boundary condition, the direct method is a little more difficult to implement.

**Collocation**

The step from the integral equation to the linear system of equations, as discussed earlier, is carried out by applying an integral equation method to an equation such as (4) to give an equation like (6). There are a range of methods for carrying this out but the most favoured technique is that of collocation, because of its inherent simplicity.

Collocation can be applied in a remarkably elementary form, which is termed \( C^{-1} \) collocation in this text since it is derived by approximating the boundary functions by a constant on each panel. In this subsection the \( C^{-1} \) collocation method is briefly outlined.

To begin with the boundary \( S \) is assumed to be expressed as a set of panels

\[ S \approx S = \sum_{j=1}^n \Delta S_j , \]

(8)

Usually the panels have a characteristic form and cannot represent a given boundary exactly. Often the \( S_j \) are referred to as elements in other texts. However, the term element refers not only to the geometry of \( \Delta S_j \) but also to the method of representing the boundary functions on \( \Delta S_j \). The \( C^{-1} \) collocation method involves representing the boundary function by a constant on each panel. For example

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\[ \varphi(p) \approx \varphi_j, \ v(p) \approx v_j \ (p \in \Delta \tilde{S}_j) \]  

The substitution of representations of this form for the boundary functions in the integral equation reduces it to discrete form. The combination of the representation of the panels and the approximation of the boundary functions, as typified by (9), defines the element.

The simplifications allow us to re-write equation (4) as the approximation

\[
\sum_{j=1}^{n} \left( M + \frac{1}{2} I \right) e_{\Delta \tilde{S}_j} (p) \varphi_j = \sum_{j=1}^{n} \{ L e \}_{\Delta \tilde{S}_j} (p) v_j \ (p \in \tilde{S})
\]

where \( e \) is the unit function \((e \equiv 1)\). The \( \{ L e \}_{\Delta \tilde{S}_j} (p) \) for example, for a specific point \( p \), are the numerical values of definite integrals and are termed the discrete form of the \( L \) integral operator.

The constant approximation is taken to be the value of the boundary functions at the representative central point (the collocation point) on each panel. By finding the discrete forms of the relevant integral operators for all the collocation points, a system of the form

\[
\sum_{j=1}^{n} \left( M + \frac{1}{2} I \right) e_{\Delta \tilde{S}_j} (p) \varphi_j \approx \sum_{j=1}^{n} \{ L e \}_{\Delta \tilde{S}_j} (p) v_j \ (p \in \tilde{S})
\]

for \( i = 1, 2, \ldots, n \) is obtained by putting \( p = p_i \) in the previous approximation. Note that because of the approximation of the boundary functions (and also the boundary approximation, if applicable), the discrete equivalent of equation (4) is an approximation relating the exact values of the boundary functions at the collocation points.

This system of approximations can now be written in the matrix-vector form

\[(M + \frac{1}{2} I)\varphi \approx Lv\]

with the matrix components defined by \([L]_{ij} = \{ L e \}_{\Delta \tilde{S}_j} (p)\), \([L]_{ij} = \{ L e \}_{\Delta \tilde{S}_j} (p)\).

The vectors \( \varphi \) and \( v \) represent the exact values of \( \varphi \) and \( v \) at the collocation points. The approximate relationship between the exact values can also be interpreted as an exact relationship between approximate values, which can be solved. The discrete forms are definite integrals that need to be computed usually by numerical integration\(^{13}\).

This document is adapted from section 1.2 of The BEM in Acoustics\(^ {14}\).

\(^{13}\) Numerical Integration
\(^{14}\) The BEM in Acoustics